

A TROPICAL ANALOG OF DESCARTES' RULE OF SIGNS

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ABSTRACT. We prove that for any degree d , there exist (families of) finite sequences $\{\lambda_{k,d}\}_{0 \leq k \leq d}$ of positive numbers such that, for any real polynomial P of degree d , the number of its real roots is less than or equal to the number of the so-called essential tropical roots of the polynomial obtained from P by multiplication of its coefficients by $\lambda_{0,d}, \lambda_{1,d}, \dots, \lambda_{d,d}$ respectively. In particular, for any real univariate polynomial $P(x)$ of degree d with non-vanishing constant term, we conjecture that one can take $\lambda_{k,d} = e^{-k^2}$, $k = 0, \dots, d$. The latter claim can be thought of as a tropical generalization of Descartes' rule of signs. We settle this conjecture up to degree 4 as well as a weaker statement for arbitrary real polynomials. Additionally we describe an application of the latter conjecture to the classical Karlin problem on zero-diminishing sequences.

1. INTRODUCTION

The famous Descartes' rule of signs claims that the number of positive roots of a real univariate polynomial does not exceed the number of sign changes in its sequence of coefficients. In what follows, among other things, we suggest a conceptually new conjectural upper bound on the number of real roots of real univariate polynomial applicable in the situation when Descartes' rule of signs gives a trivial restriction.

Recall from the literature that a sequence $\lambda = \{\lambda_k\}_{k=0}^\infty$ of real numbers is called a *multiplier sequence (of the first kind)* if the diagonal operator $T_\lambda: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ defined by $x^k \mapsto \lambda_k x^k$, for $k = 0, 1, \dots$, and extended to $\mathbb{R}[x]$ by linearity, preserves the set of real-rooted polynomials, see e.g., [CC04]. To formulate our results, we need to introduce tropical analogs of multiplier sequences. The following notion is borrowed from the classical Wiman–Valiron theory, see e.g., [Hay74]. A non-negative integer k is called a *central index* of a polynomial

$$P(x) = \sum_{i=0}^d a_i x^i$$

if there exists a real number $x_k \geq 0$ such that

$$|a_k| x_k^k \geq \sum_{i \neq k} |a_i| x_k^i. \quad (1)$$

Condition (1) has also reappeared in the context of amoebas, see, e.g., [Rul03].

To relate property (1) to real-rootedness of univariate polynomials, we recall that a real-rooted polynomial P is called *sign-independently real-rooted* if each polynomial obtained by an arbitrary sign change of the coefficients of $P(x)$ is real-rooted as well, see [PRS11]. One can easily show the following statement.

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Proposition 1. *A real polynomial P of degree d is sign-independently real-rooted if and only if every integer $k = 0, \dots, d$ is a central index of P .*

To proceed, we will need the following similar notion. A non-negative integer k is said to be a *tropical index* of P if there exists a number $x_k \geq 0$ such that

$$|a_k|x_k^k \geq \max_{i \neq k} |a_i|x_k^i. \quad (2)$$

Notice that (2) is an analog of (1) if the right-hand side of (1) is interpreted as a tropical sum. We will say that a polynomial P of degree d is *tropically real-rooted* if each integer $k = 0, \dots, d$ is a tropical index of f .

By the (standard) *tropicalization* of a real polynomial $P(x) = \sum_{i=0}^d a_i x^i$ we mean the tropical polynomial given by:

$$tr_P(\xi) = \max_{0 \leq i \leq d} (i\xi + \ln |a_i|), \quad \xi \in \mathbb{R}. \quad (3)$$

(In the literature the function $tr_P(\xi)$ is also referred to as the *Archimedean tropical polynomial* associated to P .) If $a_i = 0$, then the corresponding term in $tr_P(\xi)$ should be interpreted as $-\infty$, and thus it can be ignored when taking the maximum.

Remark 2. One can describe $tr_P(\xi)$ as follows. Define the set of points on (u, v) -plane corresponding to the monomials of P as $A_P = \{(k, \log |a_k|), k = 0, \dots, d\}$. Let $A_P(u)$ be a piecewise linear continuous function on $[0, d]$, linear on intervals $[k, k+1]$ and such that $A_P(k) = \log |a_k|$ for $k = 0, \dots, d$. Denote by $\tilde{A}_P(u)$ the least concave majorant of $A_P(u)$ on $[0, d]$, and let

$$\mathcal{NA}_P = \{v \leq \tilde{A}_P(u), 0 \leq u \leq d\}$$

be the Archimedean Newton polytope of P . Then k is a tropical index of P if and only if $(k, \log |a_k|)$ is a boundary point of \mathcal{NA}_P . Then $tr_P(\xi) = \max_{p \in \mathcal{NA}_P} (\xi, 1) \cdot p$, i.e., $tr_P(\xi)$ is the support function of \mathcal{NA}_P . Alternatively, $tr_P(\xi)$ is the Legendre transform of $-\tilde{A}_P(u)$.

Any corner of the graph of $tr_P(\xi)$, i.e., a value of ξ at which its slope changes, is called a *tropical root* of $tr_P(\xi)$. We define the (tropical) *multiplicity* of a tropical root ζ of tr_P to be one less than the number of terms of (3) for which the maximum in the right-hand side of (3) is attained at ζ . (Notice that this definition differs from the standard definition of root multiplicity in tropical geometry. This illustrates our focus on real rather than complex-valued polynomials.) With our definition of tropical root multiplicity, the number of tropical roots of $tr_P(\xi)$ counted with multiplicities is one less than the number of tropical indices of P . In particular, the number of tropical roots of $tr_P(\xi)$ is at most by one less than the number of monomials of P , which is analogous to the fact that the number of real roots of P is at most one less than its number of monomials.

We will now define positive and negative tropical roots of P using the signs of its coefficients. Let $k_0 \leq k_1 \leq \dots \leq k_m$ be the tropical indices of P . Consider two sequences $\{\text{sgn}(a_{k_i})\}_{0 \leq i \leq m}$ and $\{\text{sgn}((-1)^{k_i} a_{k_i})\}_{0 \leq i \leq m}$.

Consider two consecutive tropical indices k_{i-1} and k_i of the polynomial P ; to this pair we associate the tropical root $\xi_i = \ln(a_{i-1}/a_i)$ of $tr_P(\xi)$. If the difference $k_{i+1} - k_i$ is odd, then the pair (k_{i-1}, k_i) contributes a sign alternation in exactly one of the above sequences. In this case, we will say that ξ_i is a *positive* (respectively *negative*) *essential tropical root* of P . If the difference $k_{i+1} - k_i$ is even, then either the pair (k_{i-1}, k_i) does not contribute a sign alternation in any of the above sequences, or it contributes a sign alternation in both. In the former case we will say that ξ_i is a *non-essential tropical root* of P , and in the latter case we will say that ξ_i is a *positive-negative essential tropical root* of P . By the number of *positive essential*

tropical roots of P we mean the sum of the number of positive and positive-negative tropical roots of P . Analogously, by the number of *negative essential tropical roots of P* we mean the sum of the number of negative and positive-negative tropical roots of P . Finally by the total number of *essential tropical roots of P* we call the sum of the above two numbers.

It is easy to see that the number of essential tropical roots of P is at most d .

Example 3. Consider $P_1(x) = 1 + x^2$. The tropical indices of P_1 are $k_0 = 0$ and $k_1 = 2$. As $\ln|a_1| = \ln|0| = -\infty$, the polynomial P_1 has (with our definition of multiplicity) exactly one simple tropical root. To count the number of positive and negative tropical roots of P_1 we need to count the number of sign alternations in the sequences $\{1, 1\}$ and $\{1, (-1)^2\} = \{1, 1\}$ respectively. That is, the number of essential tropical roots of P is equal to 0.

Consider now the polynomial $P_2(x) = 1 - x^2$. Similarly to P_1 , the polynomial P_2 has one tropical root. However, to count the number of positive and negative tropical roots of P_2 we count the number of sign alternations in the sequences $\{1, -1\}$ and $\{1, -(-1)^2\} = \{1, -1\}$ respectively. That is, the number of essential tropical roots of P_2 is equal to 2.

As the definitions of the central and the tropical indices only depend on the modulus $|a_i|$, for $i = 0, \dots, d$, they immediately extend to complex-valued polynomials. However, below we restrict ourselves only to real polynomials and positive sequences λ .

A sequence $\lambda = \{\lambda_k\}_{k=0}^\infty$ is called *log-concave* if $\lambda_k^2 \geq \lambda_{k-1}\lambda_{k+1}$ for all k . In [PRS11] using discriminant amoebas, it is proven that the diagonal operator $T_\lambda: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ preserves the set of sign-independently real-rooted polynomials if and only if λ is log-concave. For this reason, log-concave sequences were called *multiplier sequences of the third kind* in *loc. cit.* We prefer to refer to log-concave sequences λ as *tropical multiplier sequences*.

Definition 4. A positive sequence $\lambda = \{\lambda_k\}_{k=0}^\infty$ is said to be a *tropical (resp. central) index preserver* if for each polynomial P the set of tropical (resp. central) indices of P is a subset of the set of tropical (resp. central) indices of the polynomial $T_\lambda[P]$.

Our first result is as follows.

Theorem 5. *For positive sequences λ , the following three conditions are equivalent:*

- (1) λ is log-concave, i.e. λ is a tropical multiplier sequence;
- (2) λ is a tropical index preserver;
- (3) λ is a central index preserver.

In particular, Theorem 5 provides an alternative (and elementary) way to settle [PRS11, Theorem 1] as requested in Problem 2 of *loc. cit.*

Corollary 6. *A positive sequence λ preserves the set of sign-independently real-rooted polynomials if and only if it is log-concave.*

In what follows, we will need a slightly more general definition of a tropicalization of P . Given an arbitrary triangular sequence $\lambda = \{\lambda_{k,j}\}_{0 \leq k \leq j, j \in \mathbb{N}}$ of positive numbers, and a univariate polynomial $P(x) = \sum_{i=0}^d a_i x^i$ of any degree d , we define its λ -tropicalization as

$$\text{tr}_P^\lambda(\xi) = \max_{0 \leq k \leq d} (k\xi + \ln|a_k| + \ln \lambda_{k,d}), \quad \xi \in \mathbb{R}. \quad (4)$$

Remark 7. Here is another description of $\text{tr}_P^\lambda(\xi)$. Let $\Theta_d(u)$ be a continuous piecewise linear function on $[0, d]$, linear on intervals $[k, k+1]$ for $k = 0, \dots, d-1$ and such that $\Theta_d(k) = \log \lambda_{k,d}$ for $k = 0, \dots, d$. Define $A_P^\lambda(u)$ as the least concave majorant of $A_P(u) + \Theta_d(u)$. Then tr_P^λ is the Legendre transform of $-A_P^\lambda(u)$, see Remark 2.

Definition 8. A finite sequence $\{\lambda_{k,d}\}_{0 \leq k \leq d}$, of positive numbers is called a *degree d (positive) real-to-tropical root preserver* if for any polynomial P of degree d (with positive coefficients), the number of essential tropical roots of (4) is greater than or equal to the number of non-zero real roots of P . A triangular sequence $\lambda = \{\lambda_{k,j}\}_{0 \leq k \leq j, j \in \mathbb{N}}$ is called a *(positive) real-to-tropical root preserver* if for each d its finite subsequence $\{\lambda_{k,d}\}_{0 \leq k \leq d}$, is a degree d (positive) real-to-tropical root preserver.

We recall that the *recession cone* of a set $X \subset \mathbb{R}^{d+1}$ is the largest pointed (i.e. including the origin) cone $C \subseteq \mathbb{R}^{d+1}$ such that if $x \in X$ then $x + c \in X$ for all $c \in C$. Our main result is as follows.

Theorem 9. *The set $\Lambda_d \subset \mathbb{R}_+^{d+1}$ (respectively $\Lambda_d^+ \subset \mathbb{R}_+^{d+1}$) of all degree d (positive) real-to-tropical root preservers $\{\lambda_{k,d}\}_{0 \leq k \leq d}$ is a nonempty closed full-dimensional subset of \mathbb{R}_+^{d+1} . Moreover, the recession cone of its logarithmic image $\text{Ln}(\Lambda_d)$ (respectively $\text{Ln}(\Lambda_d^+)$) coincides with the cone of all concave sequences of length $d+1$. (Here for any $\Omega \subset \mathbb{R}_+^k$, by $\text{Ln}(\Omega)$ we mean the set in \mathbb{R}^k obtained by taking natural logarithms of points from Ω coordinatewisely.)*

Theorem 9 shows that there exist large families of real-to-tropical root preservers in each degree, and therefore large families of real-to-tropical root preserving triangular sequences.

First we show that, if $\lambda = \{\lambda_{k,d}\}_{0 \leq k \leq d}$ is sufficiently log-concave, then λ is a degree d real-to-tropical root preserver:

Theorem 10. *Assume that a sequence $\lambda = \{\lambda_{k,d}\}_{0 \leq k \leq d}$ of positive numbers satisfies the condition:*

$$\log \frac{\lambda_{k,d}^2}{\lambda_{k-1,d} \lambda_{k+1,d}} > 2\Delta_d := \frac{d^2}{4} \log 36d + (d+1) \log d + \log 4, \quad 1 \leq k \leq d-1. \quad (5)$$

Then, for any real polynomial P , the number of positive (negative) tropical roots of tr_P^λ is greater than or equal to the number of positive (negative) roots of P . In particular, λ is a real-to-tropical root preserver.

Next we show that to be a real-to-tropical root preserver, the sequence $\lambda = \{\lambda_{k,d}\}_{0 \leq k \leq d}$ should be sufficiently log-concave.

Theorem 11. *There exists $c > 0$ with the following property. Assume that for some $k < d - 100$*

$$\log \frac{\lambda_{j,d}^2}{\lambda_{j-1,d} \lambda_{j+1,d}} < 2c, \quad j = k, \dots, k+100. \quad (6)$$

Then there exists a polynomial P of degree d with positive coefficients such that tr_P^λ has three tropical roots, and P has four negative roots. In particular, $\{\lambda_{k,d}\}_{0 \leq k \leq d}$ cannot be a degree d (positive) real-to-tropical root preserver.

In this direction, we present the following tantalizing conjecture. Consider the sequence λ^\dagger given by

$$\lambda_k^\dagger := e^{-k^2}, \quad k = 0, 1, \dots$$

we will denote by $tr_P^\dagger(\xi)$ the corresponding tropical polynomial associated to any real polynomial P , i.e.

$$tr_P^\dagger(\xi) = \max_{0 \leq k \leq d} (k\xi + \ln |a_k| - k^2), \quad \xi \in \mathbb{R}. \quad (7)$$

Conjecture 12 (Conjectural tropical analog of Descartes' rule of signs). *For any real univariate polynomial $P(x)$, the number of its positive (negative) roots does not exceed the number of positive (negative) essential tropical roots of $tr_P^\dagger(\xi)$.*

We have the following partial result supporting Conjecture 12.

Proposition 13. *Conjecture 12 holds for $d \leq 4$.*

Besides the fact that Conjecture 12 looks quite appealing, it might also shed light on possible extensions of the classical Newton inequalities for polynomials with a non-maximal number of real roots and positive coefficients. Additionally, (if settled) it also gives interesting consequences in the classical Karlin problem on zero-diminishing sequences, see [Ka68] and § 5.

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2. INTRODUCTORY RESULTS AND THEOREM 9

We will begin with the following statement. Given a sequence $\lambda = \{\lambda_k\}_{k=0}^\infty$, define its *symbol* as the formal series $S_\lambda(x) := \sum_{k=0}^\infty \lambda_k x^k$. Define its d -th truncation as $S_\lambda^{\{d\}}(x) := \sum_{k=0}^d \lambda_k x^k$.

Lemma 14. *A positive sequence λ is log-concave if and only if, for each d , the d -th truncation $S_\lambda^{\{d\}}(x)$ is a tropically real-rooted polynomial.*

Proof of Lemma 14. Assume first that λ is log-concave. For each $m \geq 1$, set $x_m := \sqrt{\lambda_{m-1}/\lambda_{m+1}}$. Then,

$$\frac{x_{m+1}}{x_m} = \frac{\lambda_m}{\sqrt{\lambda_{m-1}\lambda_{m+1}}} \frac{\lambda_{m+1}}{\sqrt{\lambda_m\lambda_{m+2}}} \geq 1,$$

so that $\{x_m\}_{m=1}^\infty$ is a non-decreasing sequence of positive real numbers. Furthermore,

$$\frac{\lambda_m x_m^m}{\lambda_{m-1} x_m^{m-1}} = \frac{\lambda_m x_m^m}{\lambda_{m+1} x_m^{m+1}} = \frac{\lambda_m}{\sqrt{\lambda_{m-1}\lambda_{m+1}}} \geq 1.$$

Since both binomials $\lambda_k x^k - \lambda_{k+1} x^{k+1}$ and $\lambda_k x^k - \lambda_{k-1} x^{k-1}$ have exactly one positive real root, we conclude that $\lambda_k x_m^k \geq \lambda_{k+1} x_m^{k+1}$ if $k \geq m$ and that $\lambda_k x_m^k \geq \lambda_{k-1} x_m^{k-1}$ if $k \leq m$. Hence,

$$\lambda_m x_m^m \geq \max_{k \neq m} \lambda_k x_m^k.$$

For the converse, assume that λ is not log-concave. That is, there exists an index m for which $\lambda_m^2 < \lambda_{m-1}\lambda_{m+1}$. Then, for $x \geq 0$,

$$\lambda_m x^m < \sqrt{\lambda_{m-1} x^{m-1} \lambda_{m+1} x^{m+1}} \leq \max(\lambda_{m-1} x^{m-1}, \lambda_{m+1} x^{m+1}).$$

In particular, m is not a tropical index of $S_\lambda(x)$. \square

Proof of Theorem 5. Let us first prove that a sequence λ is log-concave if and only if it is a tropical index preserver. Assume first that λ is log-concave. Let m be a tropical index of P , and let $x_m \geq 0$ be such that

$$a_m x_m^m \geq \max_{k \neq m} a_k x_m^k.$$

By Lemma 14 we can find ζ_m such that

$$\lambda_m \zeta_m^m \geq \max_{k \neq m} \lambda_k \zeta_m^k.$$

Then

$$\lambda_m a_m (z_m \zeta_m)^m = \lambda_m x_m^m a_m \zeta_m^m \geq \lambda_k x_m^k a_k \zeta_m^k$$

for all k . Hence, m is a tropical index of $T_\lambda[P]$. For the converse, it suffices to consider the sequence of polynomials $1 + x + \dots + x^d$, which are tropically real-rooted for all d , and use Lemma 14.

Let us now prove that λ is log-concave if and only if it is a central index preserver. Assume first that λ is log-concave, and let ζ_m be as in the proof of Theorem 5. Let m be a central index of P , and let x_m be such that

$$a_m x_m^m \geq \sum_{k \neq m} a_k x_m^k.$$

Then,

$$\lambda_m a_m (x_m \zeta_m)^m \geq \sum_{k \neq m} \lambda_m \zeta_m^m a_k x_m^k \geq \sum_{k \neq m} \lambda_k \zeta_m^k a_k x_m^k,$$

implying that m is a central index of $T_\lambda[P]$. For the converse, assume that $\lambda_m^2 < \lambda_{m-1} \lambda_{m+1}$, and consider the action of T_λ on the trinomial $x^{m-1} + 2x^m + x^{m+1}$. \square

Using Lemma 14, we can rephrase Theorem 5 in a manner similar to the classical result of Pólya and Schur, see [PS14]. Given a sequence λ of real numbers, we say that its symbol $S_\lambda(x)$ is *tropically real-rooted* if for each $d = 0, 1, \dots$, the d -th truncation $S_\lambda^{\{d\}}(x)$ is tropically real-rooted.

Corollary 15. *A positive sequence λ is a central index and tropical index preserver if and only if its symbol $S_\lambda(x)$ is tropically real-rooted.* \square

Proof of Proposition 1. To prove the *only if*-part, consider the polynomial

$$Q(x) = |a_m| x^m - \sum_{k \neq m} |a_k| x^k,$$

for some $1 \leq m \leq d-1$. Notice that Q is obtained from P by flipping signs of the coefficients and hence, by assumption, Q is real-rooted. In particular, Q has exactly two positive roots (counted with multiplicity). Let x_m be the mean value of the positive roots of Q . Then,

$$|a_m| x_m^m - \sum_{k \neq m} |a_k| x_m^k \geq 0,$$

with equality if and only if Q has a positive root of multiplicity two. In particular, m is a central index of P .

For the *if*-part, choose arbitrary signs of the coefficients of P . We note that condition (1) implies that

$$\operatorname{sgn}(P(x_m)) = \operatorname{sgn}(a_m x_m^m) = \operatorname{sgn}(a_m),$$

for $x > 0$. Using additionally Descartes' rule of signs, we conclude that the number of positive roots of P is equal to the number of sign changes in the sequence $\{a_k\}_{0 \leq k \leq d}$. Similarly, the number of negative roots of P is equal to the number of sign changes in the sequence $\{(-1)^k a_k\}_{0 \leq k \leq d}$. As $a_k \neq 0$ for each k , these two numbers sum up to d , implying that $P(x)$ is real-rooted. Since the signs of the coefficients were chosen arbitrary, we are done. \square

Proof of Corollary 6. It follows from Proposition 1 that a positive sequence λ preserves the set of sign-independently real-rooted polynomials if and only if it preserves central indices. Additionally, it follows from Theorem 5 that a positive sequence λ preserves central indices if and only if it is log-concave. \square

Proof of Theorem 9. As we are only concerned with the number of (real) roots of the polynomial P , we can consider P up to a non-vanishing scalar, i.e., we identify P with its coefficient vector $(a_0 : \dots : a_d) \in \mathbb{RP}^d$.

Let us first show that the set Λ_d is nonempty. Let $\lambda = \{\lambda_k\}_{0 \leq k \leq d}$ be a finite positive strictly log-concave sequence. By Lemma 14 we have that $S_\lambda^{\{d\}}(x)$ is tropically real-rooted. Moreover it follows from the proof of Lemma 14 and the strict log-concavity that all the tropical roots of $S_\lambda^{\{d\}}(x)$ are of multiplicity one.

Firstly, for each $P \in \mathbb{RP}^d$, we claim that there exists a positive number $s = s(P)$ such that $tr_P^{\lambda^s}(\xi)$ has at least as many distinct negative tropical roots as the number of negative roots of P . Here, λ^s denotes the sequence $\{\lambda_k^s\}_{0 \leq k \leq d}$. To prove this, notice first that, by using the change of variables $\xi \mapsto s\xi$, the number of negative tropical roots of

$$tr_P^{\lambda^s}(\xi) = \max_{0 \leq k \leq d} (k\xi + \ln |a_k| + s \ln \lambda_k)$$

is equal to the number of negative tropical roots of the tropical polynomial

$$\max_{0 \leq k \leq d} \left(s \left(k\xi + \frac{\ln |a_k|}{s} + \ln \lambda_k \right) \right),$$

the latter being equal to the Descartes' bound on the maximal number of negative roots of P , for s sufficiently big. Indeed, for all $a_k \neq 0$ the term $(\ln |a_k|)/s$ tends to 0 as $s \rightarrow \infty$.

Secondly, we claim that $s = s(P)$ can be chosen in such a way that there exists a neighborhood $N(P) \subset \mathbb{RP}^d$ of P such that for each $Q \in N(P)$ the number of negative essential tropical roots of $tr_Q^{\lambda^s}$ is not less than the number of negative roots of Q . Consider first the case $a_0 \neq 0$. Then, there is a neighborhood $N_1(P)$ of P such that the number of negative roots of $Q \in N_1(P)$ is at most equal to the number of negative roots of P . Since all negative tropical roots of $tr_P^{\lambda^s}$ are distinct, there is a neighborhood $N_2(P)$ such that the number of negative tropical roots of $tr_P^{\lambda^s}$ is equal to the number of negative tropical roots of $tr_Q^{\lambda^s}$ for all $Q \in N_2(P)$. (If P has some vanishing coefficients, then $N_2(P)$ can be chosen so that the corresponding indices are not tropical indices of Q for any $Q \in N_2(P)$.) In this case we can take $N(P) = N_1(P) \cap N_2(P)$. Complementarily, consider the case $a_0 = 0$. For each polynomial Q , let Q' denote the polynomial obtained by removing the constant term of Q . Using an inductive argument, we can choose a neighborhood $N(P)$ of P such that, for each $Q \in N(P)$, the number of negative tropical roots of $tr_Q^{\lambda^s}$ is not less than the number of negative roots of Q' . Notice that for the first non-zero coefficient a_k of P , k is a tropical index of P . If $(-1)^k a_k$ is positive, then the number of negative real roots of P increases by one if a_0 is perturbed by a small negative number, and similarly the number of negative tropical roots is increased by one, and vice versa.

Finally, to see that Λ_d is nonempty, we note that \mathbb{RP}^d is compact. Therefore, the open covering $\cup_{P \in \mathbb{RP}^d} N(P)$ of \mathbb{RP}^d has a finite subcovering $\mathbb{RP}^d \subset N(P_1) \cup \dots \cup N(P_M)$. Let $s^* = \max_{1 \leq i \leq M} s(P_i)$. Since $\lambda^{s^* - s(P_i)}$ is log-concave, it is a tropical index preserver by Theorem 5. Hence, we conclude that $\lambda^{s^*} \in \Lambda_d$.

Let us now prove that the recession cone C of $\text{Ln}(\Lambda_d)$ is equal to the set log-concave sequences of length $d+1$. The fact that the latter set is contained in C follows immediately from Theorem 5, as each log-concave sequence is a tropical index preserver. Conversely, if λ is not log-concave, then the d -th truncation $S_\lambda^{\{d\}}$

of its symbol is not tropically real-rooted. Let P be a tropically real-rooted polynomial, and let λ^* be a log-concave sequence. By a similar argument as above, we can conclude by letting s tend to infinity, that the tropical polynomial

$$tr_P^{\lambda^* \lambda^s}(\xi) = \max_{0 \leq k \leq d} (k\xi \ln |a_k| + \ln \lambda_k^* + s \ln \lambda_k)$$

is not tropically real-rooted. Hence, λ is not contained in the recession cone of the set $\text{Ln}(\Lambda_d)$.

The remaining statements of Theorem 9 follow easily from the above facts. \square

3. THEOREMS 10 AND 11

To settle Theorem 10, recall the following statement proved in e.g., [NoSh15].

Lemma 16. *For a given real polynomial P and real $x \neq 0$, assume that all tropical roots of tr_P are more than $\log 3$ away from $-\log |x|$. Let k be the tropical index corresponding to x . Then k is a central index. In particular, $P(x) \neq 0$.*

Proof. If k is the tropical index corresponding to $-\log |x|$ then $|a_j x^j| < 3^{|k-j|} |a_k x^k|$. Summing over all $j \neq k$, we get $|a_k x^k| > \sum_{j \neq k} |a_j x^j|$ and the claim follows. \square

Corollary 17. *Let P be a polynomial of degree d and assume that every integer $k = 0, \dots, d$ is a tropical index of tr_P . Assume that the tropical roots of tr_P are all simple and more than $2 \log 3$ separated one from another. Then P is sign-independently real rooted.*

Proof. Indeed, for $x = \sqrt{a_{k-1}/a_{k+1}}$ the conditions of Lemma 16 are satisfied, so k is a central index and the claim follows from Proposition 1. \square

Our proof of Theorem 10 requires two steps. At first, we prove in Lemma 20 that if a polynomial $P = \dots + a_m x^m + \dots + a_n x^n + \dots$ is a small perturbation of a polynomial $a_m x^m + \dots + a_n x^n$ with positive coefficients then it has no roots on some positive interval, with explicit bounds on the dependence of the size of the perturbation on the size of the interval.

Then we group the tropical roots of $tr_P(\xi)$ into several clusters of closely located roots and prove that in some neighborhood of each cluster the number of logarithms of positive roots of P is less than or equal to the number of positive tropical roots of tr_P in this cluster, using a generalization of Rolle's theorem presented in Lemma 21. A similar fact holds for negative roots as well.

Lemma 18. *Let P be a real polynomial and let $U = [\alpha', \alpha'']$ be a real interval such that*

- (1) *tr_P has a unique tropical root $\alpha \in U$ corresponding to two monomials $a_m x^m$ and $a_n x^n$, $m < n$, i.e., $\alpha = \frac{\log |a_n| - \log |a_m|}{n-m}$,*
- (2) *α', α'' are located more than $\log 4$ away from all tropical roots of tr_P ,*
- (3) *for all l , $m < l < n$,*

$$\log |a_l| \leq v(l) - \log d - \log 4, \tag{8}$$

where $v(u) = \alpha u + \beta$ is the unique linear function whose graph passes through $(m, \log |a_m|)$ and $(n, \log |a_n|)$.

Then P has the same number of real roots on the interval $[e^{\alpha'}, e^{\alpha''}]$ as $a_m x^m + a_n x^n$, and the same holds on the interval $[-e^{\alpha''}, -e^{\alpha'}]$.

Proof. The sum $\sum_{k < m} |a_k x^k|$ is less than $\frac{1}{3} |a_m x^m|$ on $\{x \in \mathbb{C}, \log |x| > \alpha'\}$, compare to the proof of Lemma 16. Similarly, $\sum_{k > n} |a_k x^k| \leq \frac{1}{3} |a_n x^n|$ on $\{x \in \mathbb{C}, \log |x| < \alpha''\}$. Also, $|\sum_{m < k < n} a_k x^k| \leq \frac{1}{4} (|a_m x^m| + |a_n x^n|)$ on $\{x \in \mathbb{C}, \alpha' \leq \log |x| \leq \alpha''\}$.

Consider the case $I = [e^{\alpha'}, e^{\alpha''}]$; the case of $I = [-e^{\alpha''}, -e^{\alpha'}]$ is treated similarly. Assume first that $a_n x^n$ and $a_m x^m$ have the same signs on this interval. This means that they together dominate the sum of all other terms, and there are no zeros on I at all.

If the signs are different, choose a curvilinear rectangle Π containing I and bounded by $\{\log |x| = \alpha'\}$, $\{\log |x| = \alpha''\}$ and $\{\arg x = \pm\pi/(n-m)\}$. The inequalities above imply that $a_m x^m$ dominates the sum of all other terms on $\{\log |x| = \alpha'\}$. Similarly, $a_n x^n$ dominates the sum of all other terms on $\{\log |x| = \alpha''\}$.

Moreover, the sum $a_m x^m + a_n x^n$ dominates the sum of all other terms on $\{\log |x| \in U, |\arg x| = \pi/(n-m)\}$ as the arguments of $a_m x^m$ and $a_n x^n$ are equal there. In other words, the increment of the argument of P on the boundary of Π is the same as that of $a_m x^m + a_n x^n$. Therefore P has a unique root in Π , which is necessarily real. \square

Corollary 19. *Assume that the tropical roots of tr_P are at least $2 \log 4$ apart from one another. Assume also that for any l lying between two consecutive tropical indices m, n , inequality (8) is satisfied. Then the number of positive (resp. negative) roots of P is equal to the number of positive (resp. negative) tropical roots of P .*

We will need a more refined version of Lemma 18 to take into account the signs of tropical roots.

Lemma 20. *Let P be a real polynomial and let $m < n$ be its two tropical indices with $a_m, a_n > 0$. Let $U = [\alpha', \alpha'']$ be a real interval such that*

- (1) *the tropical index of any $u \in U$ lies in $[m, n]$ and U is more than $\log 4$ away from the tropical roots of tr_P corresponding to the edges of $\tilde{A}_P(u)$ lying outside of $[m, n]$,*
- (2) *for all l , $m < l < n$, we have that either $a_l > 0$ or*

$$\log |a_l| \leq v(l) - \log d - \log 4, \quad (9)$$

where $v(u) = \alpha u + \beta$ is the linear function whose graph passes through $(m, \log |a_m|)$ and $(n, \log |a_n|)$.

Then P has no roots on $I = [e^{\alpha'}, e^{\alpha''}]$.

Proof. Let $x \in I$. As before, the sum $\sum_{k < m} |a_k x^k|$ is at most $\frac{1}{3} a_m x^m$ on I , as in the proof of Lemma 16. Similarly, $\sum_{k > n} |a_k x^k| \leq \frac{1}{3} a_n x^n$ on I . Also, $\sum'_{m < k < n} |a_k| x^k \leq \frac{1}{4} (a_m x^m + a_n x^n)$ on I , where the sum is taken over all monomials with negative coefficients. Therefore $P > 0$ on I . \square

3.1. Generalized Rolle's theorem. For a given nonnegative integer k , define the differential operator L_k by

$$L_k \left(\sum a_j x^j \right) := \sum (j - k) a_j x^j.$$

One can easily check that the latter definition is equivalent to

$$L_k(P) := x^{k+1} (x^{-k} P)'. \quad (10)$$

The following variation of Rolle's theorem immediately follows from the second definition of L_k .

Lemma 21. *Let $I \subset \mathbb{R}_+$ be some interval, then*

$$\#\{x \in I, L_k(P(x)) = 0\} \geq \#\{x \in I, P(x) = 0\} - 1.$$

One can define a natural tropical counterpart l_k of L_k as

$$l_k(\{\epsilon_j\}_{j=0}^n) = \{\text{sgn}(j-k)\epsilon_j\}_{j=0}^n,$$

where $\{\epsilon_j\}_{j=0}^n$ is any sequence of real numbers. Evidently, the number of sign changes in $\{\epsilon_j\}$ differs from that in $l_k(\{\epsilon_j\})$ by at most one.

Let α_k be the tropical roots of tr_P in the decreasing order. Let U be a connected component of the ρ -neighborhood of $\{\alpha_k\}$, where $\rho = \log 36d$.

Denote by $[m, n]$ the maximal interval such that the restriction of \tilde{A}_P to this interval has edges with slopes equal to the tropical roots of tr_P lying in U . (We can assume that $n > m + 1$ since the case $n = m + 1$ is covered by Lemma 18.)

We choose a sequence $\lambda = \{\lambda_{k,d}\}_{k=0}^d$ such that

$$\log\left(\lambda_{k-1,d}^{-1}\lambda_{k,d}^2\lambda_{k+1,d}^{-1}\right) = 2\Delta_d := \frac{d^2}{4}\log 36d + (d+1)\log d + \log 4, \quad 1 \leq k \leq d-1. \quad (10)$$

Let $q_k = (n_k, \log|a_{n_k}| + \log \lambda_{n_k})$, $k = 0, \dots, N$, be the vertices of A_P^λ on the interval $[m, n]$ in increasing order. Note that $n_0 = n$, $n_N = m$. Let $\alpha_a > \alpha_{a+1} > \dots > \alpha_b$ will be the tropical roots of tr_P lying in U .

Let $\Sigma_U = \{\text{sgn}(a_{n_k})\}$ be the sequence of signs of a_{n_k} . Choose a sequence $\{m_j\}_{j=1}^M$, $m_j \in \{n_k\}_{k=1}^{N-1}$, such that

- (i) $l_{m_1} \cdots l_{m_M}(\Sigma_U)$ has no sign changes;
- (ii) M is equal to the number of sign changes of Σ_U .

We can assume that $n > m_1 > \dots > m_{M-1} \geq m_M > m$.

Proposition 22. *The polynomial $Q = L_{m_1} \cdots L_{m_M}(P)$ has no roots in e^U .*

Proof. Without loss of generality we can take $a_n > 0$. Moreover, by rescaling of x and multiplication of P by a constant, we can assume that $a_n = |a_m| = 1$.

Let $Q = \sum_{j=0}^d b_j x^j$, $b_j = a_j \prod_{k=1}^M (j - m_k)$. We claim that Q satisfies conditions of Lemma 20.

Let us start with the first condition of Lemma 20. Let $l < m$ and

$$\kappa_{l,m}^Q = \frac{\log|a_l| + \sum_{k=1}^M \log|l - m_k| - \log|a_m| - \sum_{k=1}^M \log|m - m_k|}{l - m}$$

be the slope of the segment joining the two points in A_Q corresponding to the monomials of degree l and m . We have

$$\kappa_{l,m}^Q = \kappa_{l,m}^P - \frac{1}{m-l} \sum_{k=1}^{k_U-1} \log \frac{n_k - l}{n_k - m}. \quad (11)$$

Elementary computations show that

$$\frac{1}{m-l} \log \frac{m_k - l}{m_k - m} = \frac{1}{m_k - m} (t^{-1} \log(1+t)) \leq \frac{1}{m_k - m}, \quad t = \frac{m-l}{m_k - m} > 0,$$

as the function $t^{-1} \log(1+t)$ is monotone decreasing.

Therefore the last sum in (11) is bounded from above by $(2 + \log d)$; thus

$$\kappa_{l,m}^Q \geq \alpha_{a-1} - 2 - \log d,$$

and is more than $\log 4$ away from U , as $\rho > 2 + \log d + \log 4$. Similarly, $\kappa_{l,n}^Q \leq \alpha_{b+1} + 2 + \log d$ for $l > n$. This means that all slopes of \tilde{A}_Q to the left or to the right of $[m, n]$ are more than $\log 4$ away from U which shows that the first condition of Lemma 20 is satisfied.

To prove the second condition, we use the following elementary statement.

Lemma 23. *Let $\phi(u)$ be a continuous concave piecewise linear function on $[m, n]$ which is linear on each segment $[k, k+1]$, $k \in \mathbb{Z}$; we denote by μ_k its slope on the latter interval. Assume additionally that $\phi(m) = \phi(n) = 0$. Then,*

- (1) *if $0 \leq m_k - m_{k+1} \leq 2C$, then $\phi(u) \leq C(m-n)^2/4$;*
- (2) *if $0 \leq m_k - m_{k+1} = 2\Delta_d$, then $\phi(k) \geq (n-m-1)\Delta_d$ for all $m < k < n, k \in \mathbb{Z}$.*

Corollary 24.

$$\log |a_l| \leq \frac{d^2}{4} \log 36d, \quad m \leq l \leq n. \quad (12)$$

Proof. By definition of U , one can apply the first claim of Lemma 23 to the restriction of \hat{A}_P to the segment $[m, n]$. \square

Corollary 25. *Choose $l \in [m, n]$, $l \in \mathbb{Z}$ and $l \notin \{n_k\}$. Then*

$$\log |a_l| \leq \frac{d^2}{4} \log 36d - \Delta_d,$$

where Δ_d is the same as in Theorem 10.

Proof. Condition $l \notin \{n_k\}$ means that $\log |a_l| + \log \lambda_{l,d} < \alpha l + \beta$, where α, β are chosen in such a way that $\alpha m + \beta = \log \lambda_{m,d}$ and $\alpha n + \beta = \log \lambda_{n,d}$. Therefore

$$\log |a_l| \leq -(\Theta_d(u) - \alpha u - \beta),$$

and the bound follows from the second claim of Lemma 23 applied to $\phi(u) = \Theta_d(u) - \alpha u - \beta$. \square

Now, $\log |b_l| = \log |a_l| + \sum \log |m_k - l| \leq \log |a_l| + d \log d$. Therefore

$$\log |b_l| \leq \frac{d^2}{4} \log 36d - \Delta_d + d \log d \leq -\log d - 4,$$

which implies the second condition of Lemma 20, since both $\log |b_m|, \log |b_n|$ are positive. This finishes the proof of Proposition 22. \square

Corollary 26. *Let M be the number of sign changes in $\{a_{n_k}\}$, where $\{n_k\}$ are tropical indices of tr_P^λ on the interval $[m, n]$. Then P has at most M roots on e^U .*

Proof. Follows from Proposition 22, and Lemma 21. \square

Proof of Theorem 10. Applying Corollary 26 to each connected component of the $\log 36d$ -neighborhood of the set of tropical roots of tr_P (and using Lemma 16 outside of it), we see that the number of positive roots of P does not exceed the number of positive tropical roots of tr_P^λ .

Changing $P(x)$ to $P(-x)$, we get the same statement for the negative roots. In particular, we conclude that $\{\lambda_{k,d}\}$ defined in (10) is a real-to-tropical root preserver. \square

To prove Theorem 11, we need an auxiliary statement.

Lemma 27. *There exists a polynomial R of degree 100 with 4 simple negative roots, whose leading and constant coefficients are equal to 1 and the remaining coefficients are non-negative and strictly less than 1.*

Proof of Lemma 27. Set $Q_1(x) = x + 1$ and define $Q_{k+1}(x) = Q_k(x)(x^n + 1)$, $k = 2, 3, \dots$, where n is the smallest odd number greater than $\deg Q_k$. Note that

- (1) all coefficients of Q_k are either 1 or 0,
- (2) $Q_k(x)$ is divisible by $(x+1)^k$.

Take $Q_4(x^5)$ (which has a root of multiplicity 4 at -1), add some small positive multiple of $(x+1)^3$ to split of a simple real root from the 4-tuple root at -1 , then add an even smaller positive multiple of $(x+1)^2$ to split of another simple root from -1 , and then add an even smaller multiple of $x+1$ to split of the third simple root. (Note that $Q_4(x^5)$ has no monomials of degree 1, 2, 3.)

The resulting perturbation \tilde{Q}_4 has four negative roots, is of degree 100, has a leading term equal to 1, the constant term $a_0 > 1$, and all the remaining coefficients at most 1. (All of them are equal to either 0 or 1 except in degrees 1, 2, 3, where they are small positive numbers). Define $R = a_0^{-1}\tilde{Q}_4(a_0^{1/100}x) = x^{100} + \dots + 1$, with all other coefficients non-negative and smaller than $a_0^{-1/100}$. \square

Proof of Theorem 11. Starting with the above polynomial R , we construct a polynomial P with 4 negative roots and with only three tropical roots. Note that

$$A_R(u) \leq \tilde{A}_R(u) \equiv 0 \quad \text{for } 0 \leq u \leq 100,$$

with equality for $u = 0$ and 100 only.

Choose $c > 0$ in Theorem 11 such that $A_R(u) \leq -cu(100 - u)$ for $0 \leq u \leq 100$. Inequality (6) implies that $\Theta_d(u)$ is almost flat on the interval $[k, k+100]$, see Remark 7. More exactly, there exists a linear function $\ell(u)$ such that,

$$\Theta_d(u) \leq \ell(u) + cu(100 - u), \quad k \leq u \leq k+100,$$

with equality for $u = k, k+100$. Therefore $A_{x^k R}(u) + \Theta_d(u) \leq \ell(u)$ for $0 \leq u \leq 100$, with equality for $u = k, k+100$ (i.e., lies below its chord on $[k, k+100]$). Therefore $A_{x^k R}^\lambda(u)$ is linear, and $tr_{x^k R}^\lambda(\xi)$ has just one tropical root.

Now, choose $\delta > 0$ so small that $P = \delta(x^d + 1) + x^k R$ still has 4 negative simple roots. Then $tr_P^\lambda(\xi)$ has at most 3 tropical roots, since only two extra monomials were added. The latter choice of P settles Theorem 11. \square

4. PROPOSITION 13

We start with some explicit information about Λ_d and Λ_d^+ for small d , compare to Theorem 9.

Lemma 28. (1) For $d = 1$, $\Lambda_1^+ = \Lambda_1 = \mathbb{R}_+$;
 (2) For $d = 2$, $\Lambda_2^+ = \Lambda_2 = \{\lambda \mid 4\lambda_1^2 \geq \lambda_0\lambda_2\}$.

Proof. (1) Note that it is enough to consider only fully supported polynomials P . Then, by normalization, we can assume that $a_0 = a_1 = 1$. For $d = 1$ there is nothing to prove.

(2) For $d = 2$, consider a polynomial $P(x) = 1 + x + ax^2$. Then, $P(x)$ has two real roots if and only if $a \leq \frac{1}{4}$. If $a < 0$, then $tr_P^\lambda(\xi)$ has two essential tropical roots for all a . Thus it suffices to consider only the case $a > 0$. We need to compare the above inequality to the condition that the tropical polynomial

$$tr_P^\lambda(\xi) = \max(\ln \lambda_0, \xi + \ln \lambda_1, 2\xi + \ln a + \ln \lambda_2),$$

has two tropical roots. One can easily check that this happens if and only if $\lambda_1^2 \geq a\lambda_0\lambda_2$. This inequality holds for all $0 \leq a \leq \frac{1}{4}$ if and only if $4\lambda_1^2 \geq \lambda_0\lambda_2$. Clearly, the latter inequality is necessary and sufficient also if we restrict ourselves to polynomials with positive coefficients. \square

Lemma 29. For $d = 4$, Λ_4^+ contains the set defined by the system of inequalities:

$$\begin{cases} 2\lambda_1^2 \geq \lambda_0\lambda_2, & 9\lambda_2^2 \geq 4\lambda_1\lambda_3, & 2\lambda_3^2 \geq \lambda_2\lambda_4, \\ 2(\sqrt[4]{3}-1)\lambda_1^4 \geq \sqrt[4]{3}\lambda_0^3\lambda_4, & 2(\sqrt[4]{3}-1)\lambda_3^4 \geq \sqrt[4]{3}\lambda_0\lambda_4^3. \end{cases} \quad (13)$$

Proof. As we consider only P with positive coefficients, we can without loss of generality restrict ourselves to the case $a_0 = a_4 = 1$, i.e.

$$P(x) = 1 + a_1x + a_2x^2 + a_3x^3 + x^4.$$

We compare the appearance of its real roots with the appearance of tropical roots of the tropical polynomial

$$tr_P^\lambda(\xi) = \max(\ln \lambda_0, \xi + \ln a_1 + \ln \lambda_1, 2\xi + \ln a_2 + \ln \lambda_2, 3\xi + \ln a_3 + \ln \lambda_3, 4\xi + \ln \lambda_4),$$

where $\lambda_0, \dots, \lambda_4$ are variables. For real-rooted polynomials, we obtain the inequalities:

$$8\lambda_1^2 \geq 3\lambda_0\lambda_2, \quad 9\lambda_2^2 \geq 4\lambda_1\lambda_3, \quad 8\lambda_3^2 \geq 3\lambda_2\lambda_4.$$

Let us now consider polynomials $P(x)$ with exactly two real roots. When decreasing a_1, a_2 , and a_3 simultaneously, one can only decrease the number of essential tropical roots. Therefore it suffices to prove the statement for polynomials $P(x)$ with a real double root only. With our normalization, such a polynomial can be written as

$$\begin{aligned} P(x) &= (r+x)^2(r^{-2}+sx+x^2) \\ &= 1 + (2r^{-1}+sr^2)x + (r^{-2}+2sr+r^2)x^2 + (2r+s)x^3 + x^4. \end{aligned}$$

Associated tropical polynomials are of the form

$$\begin{aligned} tr_P(\xi) &= \max\left(\ln \lambda_0, \xi + \ln(2r^{-1}+sr^2) + \ln \lambda_1, \right. \\ &\quad \left. 2\xi + \ln(r^{-2}+2sr+r^2) + \ln \lambda_2, \right. \\ &\quad \left. 3\xi + \ln(2r+s) + \ln \lambda_3, \quad 4\xi + \ln \lambda_4\right). \end{aligned}$$

We will divide our consideration into two cases. If $r \leq 1$, then we will require that the first order term dominates the even order terms at some point. If $r \geq 1$ we will require that the third order term dominates the even order terms at some point. In the first case, we consider the point

$$\xi_1 = -\ln(2r^{-1}+sr^2) - \ln \lambda_1 + \ln \lambda_0$$

and obtain the inequalities

$$\frac{\lambda_1^2}{\lambda_0\lambda_2} \geq \frac{1+2sr^3+r^4}{(2+sr^3)^2} \quad \text{and} \quad \frac{\lambda_1^4}{\lambda_0^3\lambda_4} \geq \frac{r^4}{(2+sr^3)^4}.$$

Since we require the coefficients of P to be positive, it is sufficient that these inequalities are valid for all $0 < r \leq 1$ and $s \geq -\frac{2}{\sqrt[4]{3}}$. We find that

$$\sup_{r,s} \frac{1+2sr^3+r^4}{(2+sr^3)^2} = \sup_r \frac{1}{3-r^4} = \frac{1}{2}.$$

and that

$$\sup_{r,s} \frac{r}{2+sr^3} = \sup_r \frac{r}{2-\frac{2}{\sqrt[4]{3}}r^3} = \frac{\sqrt[4]{3}}{2(\sqrt[4]{3}-1)}.$$

Thus, in case $r \leq 1$ we obtain the inequalities

$$2\lambda_1^2 \geq \lambda_0\lambda_2 \quad \text{and} \quad 2(\sqrt[4]{3}-1)\lambda_1^4 \geq \sqrt[4]{3}\lambda_0^3\lambda_4.$$

By symmetry, for $r \geq 1$, we obtain the inequalities

$$2\lambda_3^2 \geq \lambda_2\lambda_4 \quad \text{and} \quad 2(\sqrt[4]{3}-1)\lambda_3^4 \geq \sqrt[4]{3}\lambda_0\lambda_4^3.$$

Altogether, we derived the system (13). \square

Proof of Proposition 13. Up to degree 3, the statement is covered by Lemma 28, as there is nothing to prove in the case of a cubic polynomial with one real root. The case of degree 4 follows immediately from Lemma 29. \square

5. APPLICATION TO ZERO-DIMINISHING SEQUENCES

We start with the following standard definition, see e.g., [CC80], [CC95].

Definition 30. A sequence $\Gamma = \{\lambda_k\}_{k=0}^d$ of real numbers is called a complex zero decreasing sequence in degree d (a CZDS in degree d , for short) if, for any polynomial $P = a_0 + a_1x + \dots + a_dx^d$ with real coefficients, the polynomial $T_\lambda(P) = \lambda_0a_0 + \lambda_1a_1x + \dots + \lambda_da_dx^d$ has no more non-real roots than P .

A sequence $\Gamma = \{\lambda_k\}_{k=0}^\infty$ of real numbers is called a complex zero decreasing sequence (a CZDS, for short) if for every $d \in \mathbb{N}$ the sequence $\Gamma = \{\lambda_k\}_{k=0}^d$ is a CZDS in degree d .

Laguerre's classical result from 1884 gives the so far best recipe how to generate such sequences. Namely,

Theorem 31 (p. 116 of [La84]). For any real polynomial $f(z)$ with all strictly negative roots, the sequence $\{f(n)\}$, $n = 0, 1, \dots$ is a CZDS.

On p. 382 of his well-known book [Ka68], S. Karlin posed the problem of characterizing the inverses of CZDS which are called *zero-diminishing sequences* (ZDS, for short). This problem is sometimes referred to as the Karlin problem.¹ Substantial information about CZDS can be found in section 4 of [CC96] and a number of earlier papers. Several interesting attempts to find the converse of Laguerre's theorem and to solve the Karlin problem were carried out over the years, the most successful of them apparently being [BCC01] and [BR08]. (For the history of the subject consult [CC80] and [Pi02].) But inspite of some hundred and thirty years passed since the publication of [La84] and certain partial progress, satisfactory characterization of the sets of all complex zero decreasing sequences and/or of all zero-diminishing sequences is still unavailable at present. In particular, it is still unknown whether the rapidly decreasing sequence $\{e^{-k^\alpha}\}_{k=0}^\infty$ with $\alpha > 2$ is a CZDS.

We will now illustrate how the theory developed in this paper can be applied to obtain new results regarding CZDS.

Theorem 32. Let $\lambda^* = \{\lambda_{k,j}^*\}_{0 \leq k \leq j, j \in \mathbb{N}}$ be a triangular real-to-tropical root preserver. Let $\lambda = \{\lambda_k\}_{k=0}^d$ be a sequence of positive numbers. If the set of central indices of the polynomial

$$Q_d(x) = \sum_{k=0}^d \frac{\lambda_k}{\lambda_{k,d}^*} x^k$$

is equal to $\{0, 1, \dots, d\}$, i.e., $Q_d(x)$ is sign-independently real rooted, then λ is a CZDS in degree d .

In particular, if any initial segment $\{\lambda_k\}_{k=0}^d$ of a sequence $\{\lambda_k\}_{k=0}^\infty$ satisfies this condition then $\{\lambda_k\}_{k=0}^\infty$ is a CZDS.

Proof. Consider a polynomial $P(x) = \sum_{i=0}^d a_i x^i$, and its image

$$T_\lambda[P] = \sum_{i=0}^d \lambda_i a_i x^i = \sum_{i=0}^d \frac{\lambda_i}{\lambda_{i,d}^*} \lambda_{i,d}^* a_i x^i$$

¹ In [CC80] the authors initially claimed that they have solved Karlin's problem, but later they discovered a mistake in the presented solution.

under the operator T_λ . Since λ^* is a triangular real-to-tropical root preserver, the number of essential tropical roots of the polynomial

$$R(x) = \sum_{i=0}^d \lambda_{i,d}^* a_i x^i$$

is at least equal to the number of real roots of P . Let $0 = k_0 < k_1 < \dots < k_m = d$ be the tropical indices of $R(x)$, and let $x_0, \dots, x_m > 0$ be such that the tropical index k_j is dominating at x_j , that is

$$\lambda_{j,d}^* |a_j| x_j^j \geq \max_{i \neq j} \lambda_{i,d}^* |a_i| x_j^i. \quad (14)$$

Since each k_j is a central index of the polynomial $Q_d(x)$, we can find points y_1, \dots, y_m such that

$$\frac{\lambda_j}{\lambda_{j,d}^*} y_j^j \geq \sum_{i \neq j} \frac{\lambda_i}{\lambda_{i,d}^*} y_j^i. \quad (15)$$

Inequalities (14) and (15) imply that

$$\lambda_j |a_j| (x_j y_j)^j = \frac{\lambda_j}{\lambda_{j,d}^*} y_j^j \lambda_{j,d}^* |a_j| x_j^j \geq \sum_{i \neq j} \frac{\lambda_i}{\lambda_{i,d}^*} y_j^i \lambda_{i,d}^* |a_i| x_j^i = \sum_{i \neq j} \lambda_i |a_i| (x_j y_j)^i.$$

Thus, each k_j is a central index of $T_\lambda[P]$. In particular, the number of real roots of $T_\lambda[P]$ is at least equal to the number of essential tropical roots of $R(x)$, which in turn is at least equal to the number of real roots of P . \square

Theorem 33. *Assume that the sequence $\{e^{-k^2}\}_{k=0}^\infty$ is a real-to-tropical root preserver. Then, the sequence $\{e^{-k^\alpha}\}_{k=0}^\infty$ is a CZDS for all $\alpha \geq 3$.*

Proof. For the corresponding polynomial $Q_d(x) = \sum_{k=0}^d e^{-k^\alpha + k^2} x^k$ the tropical roots are $\gamma_k = 2k - 1 + (k-1)^\alpha - k^\alpha$. We see that

$$\gamma_k - \gamma_{k+1} = -2 + (k-1)^\alpha + (k+1)^\alpha - 2k^\alpha > -2 + \alpha(\alpha-1)k^{\alpha-2}$$

as soon as $\alpha > 3$. Already for $\alpha > 2.608\dots$ and $k \geq 1$, the latter expression is bigger than $2 \log 3$. Therefore Corollary 17 implies that $Q_d(x)$ is a sign-independently real rooted for any $\alpha > 3$. Then Theorem 32 implies the result. \square

Remark 34. The lower bound $\alpha \geq 3$ for the sequence $\{e^{-k^\alpha}\}_{k=0}^\infty$ to be a CZDS is apparently not sharp. In particular, computer experiments show that conclusion of Theorem 32 holds for $\alpha > 2.437623\dots$. But since we do not currently see how to prove Conjecture 12, we were not trying to get the optional lower bound with the help of Theorem 32.

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